4.1 Sets

Name

$\neq$ – Inequality
$\notin$ – Non-membership

Definition

$$
\begin{array}{|l}
[X] \\ \hline
\_ \neq \_ : X \leftrightarrow X \\
\_ \notin \_ : X \leftrightarrow \mathbb{P}\, X \\ \hline
\forall\, x, y : X \bullet x \neq y \Leftrightarrow \neg\,(x = y) \\
\forall\, x : X;\ S : \mathbb{P}\, X \bullet x \notin S \Leftrightarrow \neg\,(x \in S)
\end{array}
$$

Description

The relations $\neq$ and $\notin$ are the complements of the equality and membership relations expressed by $=$ and $\in$ respectively.

Laws

$x \neq y \Rightarrow y \neq x$

Name

dom, ran – Domain and range of a relation

Definition

$$
\begin{array}{|l}
[X, Y] \\ \hline
\mathrm{dom} : (X \leftrightarrow Y) \longrightarrow \mathbb{P}\, X \\
\mathrm{ran} : (X \leftrightarrow Y) \longrightarrow \mathbb{P}\, Y \\ \hline
\forall\, R : X \leftrightarrow Y \bullet \\
\quad \mathrm{dom}\, R = \{\, x : X;\ y : Y \mid x \underline{R}\, y \bullet x \,\} \wedge \\
\quad \mathrm{ran}\, R = \{\, x : X;\ y : Y \mid x \underline{R}\, y \bullet y \,\}
\end{array}
$$

Name

$\lhd$ – Domain restriction
$\rhd$ – Range restriction

Definition

$$
\begin{array}{|l}
[X, Y] \\ \hline
\_ \lhd \_ : \mathbb{P}\, X \times (X \leftrightarrow Y) \longrightarrow (X \leftrightarrow Y) \\
\_ \rhd \_ : (X \leftrightarrow Y) \times \mathbb{P}\, Y \longrightarrow (X \leftrightarrow Y) \\ \hline
\forall\, S : \mathbb{P}\, X;\ R : X \leftrightarrow Y \bullet \\
\quad S \lhd R = \{\, x : X;\ y : Y \mid x \in S \wedge x \underline{R}\, y \bullet x \mapsto y \,\} \\
\forall\, R : X \leftrightarrow Y;\ T : \mathbb{P}\, Y \bullet \\
\quad R \rhd T = \{\, x : X;\ y : Y \mid x \underline{R}\, y \wedge y \in T \bullet x \mapsto y \,\}
\end{array}
$$

Name

$⩤$ – Domain anti-restriction
$⩥$ – Range anti-restriction

Definition

$$
\begin{array}{|l}
[X, Y] \\ \hline
\_ ⩤ \_ : \mathbb{P}\, X \times (X \leftrightarrow Y) \longrightarrow (X \leftrightarrow Y) \\
\_ ⩥ \_ : (X \leftrightarrow Y) \times \mathbb{P}\, Y \longrightarrow (X \leftrightarrow Y) \\ \hline
\forall\, S : \mathbb{P}\, X;\ R : X \leftrightarrow Y \bullet \\
\quad S ⩤ R = \{\, x : X;\ y : Y \mid x \notin S \wedge x \underline{R}\, y \bullet x \mapsto y \,\} \\
\forall\, R : X \leftrightarrow Y;\ T : \mathbb{P}\, Y \bullet \\
\quad R ⩥ T = \{\, x : X;\ y : Y \mid x \underline{R}\, y \wedge y \notin T \bullet x \mapsto y \,\}
\end{array}
$$

Name

$\oplus$ – Overriding

Definition

$$
\begin{array}{|l}
[X, Y] \\ \hline
\_ \oplus \_ : (X \leftrightarrow Y) \times (X \leftrightarrow Y) \longrightarrow (X \leftrightarrow Y) \\ \hline
\forall\, Q, R : X \leftrightarrow Y \bullet \\
\quad Q \oplus R = ((\mathrm{dom}\, R) ⩤ Q) \cup R
\end{array}
$$

Name

$⇸$ – Partial functions
$\longrightarrow$ – Total functions
$⤔$ – Partial injections
$\rightarrowtail$ – Total injections
$⤀$ – Partial surjections
$\twoheadrightarrow$ – Total surjections
$⤖$ – Bijections

Definition

$$X ⇸ Y == \{\, f : X \leftrightarrow Y \mid (\forall\, x : X;\ y_1, y_2 : Y \bullet (x \mapsto y_1) \in f \wedge (x \mapsto y_2) \in f \Rightarrow y_1 = y_2) \,\}$$

$$X \longrightarrow Y == \{\, f : X ⇸ Y \mid \mathrm{dom}\, f = X \,\}$$

$$X ⤔ Y == \{\, f : X ⇸ Y \mid (\forall\, x_1, x_2 : \mathrm{dom}\, f \bullet f(x_1) = f(x_2) \Rightarrow x_1 = x_2) \,\}$$

$$X \rightarrowtail Y == (X ⤔ Y) \cap (X \longrightarrow Y)$$

$$X ⤀ Y == \{\, f : X ⇸ Y \mid \mathrm{ran}\, f = Y \,\}$$

$$X \twoheadrightarrow Y == (X ⤀ Y) \cap (X \longrightarrow Y)$$

$$X ⤖ Y == (X \twoheadrightarrow Y) \cap (X \rightarrowtail Y)$$

4.2 Relations

Name

$\leftrightarrow$ – Binary relations
$\mapsto$ – Maplet

Definition

$X \leftrightarrow Y == \mathbb{P}(X \times Y)$

$$
\begin{array}{|l}
[X, Y] \\ \hline
\_ \mapsto \_ : X \times Y \longrightarrow X \times Y \\ \hline
\forall\, x : X;\ y : Y \bullet \\
\quad x \mapsto y = (x, y)
\end{array}
$$

Description

If X and Y are sets, then $X \leftrightarrow Y$ is the set of *binary relations* between X and Y. Each such relation is a subset of $X \times Y$. The 'maplet' notation $x \mapsto y$ is a graphic way of expressing the ordered pair (x, y).

The definition of $X \leftrightarrow Y$ given here repeats the one given on page 88.

Name

dom, ran – Domain and range of a relation

Definition

$$
\begin{array}{|l}
[X, Y] \\ \hline
\mathrm{dom} : (X \leftrightarrow Y) \longrightarrow \mathbb{P}\, X \\
\mathrm{ran} : (X \leftrightarrow Y) \longrightarrow \mathbb{P}\, Y \\ \hline
\forall\, R : X \leftrightarrow Y \bullet \\
\quad \mathrm{dom}\, R = \{\, x : X;\ y : Y \mid x \underline{R}\, y \bullet x \,\} \wedge \\
\quad \mathrm{ran}\, R = \{\, x : X;\ y : Y \mid x \underline{R}\, y \bullet y \,\}
\end{array}
$$

Name

$\mathbb{N}$	–	Natural numbers
$\mathbb{Z}$	–	Integers
$+, -, *, \mathsf{div}, \mathsf{mod}$	–	Arithmetic operations
$<, \leq, \geq, >$	–	Numerical comparison

Definition

$[\mathbb{Z}]$

$$\begin{array}{|l} \_ + \_, \_ - \_, \_ * \_ : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z} \\ \_ \,\mathsf{div}\, \_, \_ \,\mathsf{mod}\, \_ : \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \longrightarrow \mathbb{Z} \\ - : \mathbb{Z} \longrightarrow \mathbb{Z} \\ \_ < \_, \_ \leq \_, \_ \geq \_, \_ > \_ : \mathbb{Z} \leftrightarrow \mathbb{Z} \\ \hline \ldots \text{ definitions omitted } \ldots \end{array}$$

$\mathbb{N} == \{ n : \mathbb{Z} \mid n \geq 0 \}$

Name

min, max – Minimum and maximum of a set of numbers

Definition

$$\begin{array}{|l} min : \mathbb{P}_1 \mathbb{Z} \rightarrow\!\!\!\!\!\!+ \;\mathbb{Z} \\ max : \mathbb{P}_1 \mathbb{Z} \rightarrow\!\!\!\!\!\!+ \;\mathbb{Z} \\ \hline min = \{ S : \mathbb{P}_1 \mathbb{Z}; m : \mathbb{Z} \mid \\ \qquad m \in S \land (\forall n : S \bullet m \leq n) \bullet S \mapsto m \} \\ max = \{ S : \mathbb{P}_1 \mathbb{Z}; m : \mathbb{Z} \mid \\ \qquad m \in S \land (\forall n : S \bullet m \geq n) \bullet S \mapsto m \} \end{array}$$

Definition

$\mathrm{seq}\, X == \{ f : \mathbb{N} \rightarrow\!\!\!\!\!\!\Vdash X \mid \mathrm{dom}\, f = 1 \ldots \# f \}$

$\mathrm{seq}_1\, X == \{ f : \mathrm{seq}\, X \mid \# f > 0 \}$

$\mathrm{iseq}\, X == \mathrm{seq}\, X \cap (\mathbb{N} \rightarrowtail\!\!\!\!\!\!+ X)$

Definition

$$\begin{array}{|l} [X] \\ \_ \frown \_ : \mathrm{seq}\, X \times \mathrm{seq}\, X \longrightarrow \mathrm{seq}\, X \\ rev : \mathrm{seq}\, X \longrightarrow \mathrm{seq}\, X \\ \hline \forall s, t : \mathrm{seq}\, X \bullet \\ \qquad s \frown t = s \cup \{ n : \mathrm{dom}\, t \bullet n + \# s \mapsto t(n) \} \\ \forall s : \mathrm{seq}\, X \bullet \\ \qquad rev\, s = (\lambda n : \mathrm{dom}\, s \bullet s(\# s - n + 1)) \end{array}$$

Definition

$$\begin{array}{|l} [X] \\ head, last : \mathrm{seq}_1\, X \longrightarrow X \\ tail, front : \mathrm{seq}_1\, X \longrightarrow \mathrm{seq}\, X \\ \hline \forall s : \mathrm{seq}_1\, X \bullet \\ \qquad head\, s = s(1) \land \\ \qquad last\, s = s(\# s) \land \\ \qquad tail\, s = (\lambda n : 1 \ldots \# s - 1 \bullet s(n + 1)) \land \\ \qquad front\, s = (1 \ldots \# s - 1) \lhd s \end{array}$$

Definition

$$\begin{array}{|l} [X] \\ \_ \upharpoonleft \_ : \mathbb{P}\, \mathbb{N}_1 \times \mathrm{seq}\, X \longrightarrow \mathrm{seq}\, X \\ \_ \upharpoonright \_ : \mathrm{seq}\, X \times \mathbb{P}\, X \longrightarrow \mathrm{seq}\, X \\ squash : (\mathbb{N}_1 \rightarrow\!\!\!\!\!\!\Vdash X) \longrightarrow \mathrm{seq}\, X \\ \hline \forall U : \mathbb{P}\, \mathbb{N}_1; s : \mathrm{seq}\, X \bullet \\ \qquad U \upharpoonleft s = squash\, (U \lhd s) \\ \forall s : \mathrm{seq}\, X;\ V : \mathbb{P}\, X \bullet \\ \qquad s \upharpoonright V = squash\, (s \rhd V) \\ \forall f : \mathbb{N}_1 \rightarrow\!\!\!\!\!\!\Vdash X \bullet \\ \qquad squash\, f = f \circ (\mu\, p : 1 \ldots \# f \rightarrowtail\!\!\!\!\!\rightarrow \mathrm{dom}\, f \mid p \circ succ \circ p^{\sim} \subseteq (\_ < \_)) \end{array}$$

Name

prefix	–	Prefix relation
suffix	–	Suffix relation
in	–	Segment relation

Definition

$$\begin{array}{|l} [X] \\ \_ \,\mathsf{prefix}\, \_, \_ \,\mathsf{suffix}\, \_, \_ \,\mathsf{in}\, \_ : \mathrm{seq}\, X \leftrightarrow \mathrm{seq}\, X \\ \hline \forall s, t : \mathrm{seq}\, X \bullet \\ \qquad s \,\mathsf{prefix}\, t \Leftrightarrow (\exists v : \mathrm{seq}\, X \bullet s \frown v = t) \land \\ \qquad s \,\mathsf{suffix}\, t \Leftrightarrow (\exists u : \mathrm{seq}\, X \bullet u \frown s = t) \land \\ \qquad s \,\mathsf{in}\, t \Leftrightarrow (\exists u, v : \mathrm{seq}\, X \bullet u \frown s \frown v = t) \end{array}$$

$$\begin{array}{|l} \Delta State[X, Y] \\ State[X, Y] \\ State'[X, Y] \end{array}$$

$$\begin{array}{|l} \Xi State \\ State \\ State' \\ \hline \theta State = \theta State' \end{array}$$

Ref:

The Z Notation:

A Reference Manual

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